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Current Algebras and QP Manifolds

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Abstract

Generalized current algebras introduced by Alekseev and Strobl in two dimensions are reconstructed by a graded manifold and a graded Poisson brackets. We generalize their current algebras to higher dimensions. QP manifolds provide the unified structures of current algebras in any dimension. Current algebras give rise to structures of Leibniz/Loday algebroids, which are characterized by QP structures. Especially, in three dimensions, a current algebra has a structure of a Lie algebroid up to homotopy introduced by Uchino and one of the authors which has a bracket of a generalization of the Courant-Dorfman bracket. Anomaly cancellation conditions are reinterpreted as generalizations of the Dirac structure.

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1 Introduction

Alekseev and Strobl [1] have generalized a current algebra in two dimensions to a target space $TM \oplus T^*M$, which is described by the Courant bracket, where M is a manifold in d dimensions. The condition that currents close and does not have anomalies is geometrically characterized by the Dirac structure on $TM \oplus T^*M$. Since the current algebra naturally contains fluxes, this is related to the string theory with a flux background. Generalizations of their current algebras to higher dimensions [2], or more general currents in two dimensions [3] have been constructed.

The canonical commutation relations of canonical conjugates can be reformulated in terms of the supergeometry. A Poisson bracket is constructed from a Schouten-Nijenhuis bracket and a Poisson bivector field [4]. We generalize this formulation to current algebras.

The algebra to be closed under the Courant bracket is the Courant algebroid [5], [6]. The Courant algebroid has a supermanifold construction [7] by the derived bracket [8]. This construction is closely related to a QP structure of a topological field theory [9] in three dimensions. In fact, the supermanifold construction of the Courant bracket has derived a topological sigma model in three dimensions, which is called the Courant sigma model [10], [11]. This is a direct application of a AKSZ construction [12], [13], [11] in a topological field theory to three dimensions. The supergeometry is a key idea again.

In this paper, current algebras described in Alekseev and Strobl are reconstructed in terms of a QP structure. This reconstruction proposes the unified structure of current algebras in an arbitrary dimension. Moreover current algebras [2] in a n dimensions worldvolume are consistently generalized in order to contain more general dynamical systems. We point out that current 'algebras' have structures such as the Lie algebroids [14], the Courant algebroid and their generalizations, containing the Loday algebroids. Especially, current algebra in three dimensions has a structure of a Lie algebroid up to homotopy [15], more generally, the H-twisted Lie algebroid [16]. Anomaly cancellation conditions are clarified in terms of QP manifolds of degree n and gives rise to a generalization of the Dirac structure.

The paper is organized as follows. In section 2, a supermanifold construction of current algebras in one dimension is considered. In section 3, the paper of Alekseev and Strobl is reviewed. In section 4, mathematics related to this article is explained. In section 5, the current algebras in two dimensions are reconstructed from a QP manifold. In section 6, the

generalized current algebras in three dimensions are introduced and reinterpreted from a QP manifold. In section 7, the generalized current algebras in n dimensions are discussed. Section 8 is conclusions and discussion.

2 Current Algebras in One Dimension

First we consider one dimensional worldline case as a typical and well known example of our ideas. Let us consider a space M of d dimensions defined on one dimensional worldline $X_1 = \boldsymbol{R}$, which possesses the canonical conjugates (x^I, p_J) on a phase space T^*M, where the Poisson brackets are given by

$$\{x^I, x^J\}_{P.B.} = 0, \quad \{x^I, p_J\}_{P.B.} = \delta^I{}_J, \quad \{p_I, p_J\}_{P.B.} = 0, \tag{2.1}$$

where $I, J, \cdots$ are indices on M. Introducing a gauge potential $A_I(x)$ on M, the canonical momentum p_I is shifted as $p_I \to p_I + A_I$. The Poisson brackets of canonical conjugates are twisted by a closed 2-form $H_{IJ} = \partial_I A_J - \partial_J A_I$ on the phase space as follows:

$$\{x^I, x^J\}_{P.B.} = 0, \quad \{x^I, p_J\}_{P.B.} = \delta^I{}_J, \quad \{p_I, p_J\}_{P.B.} = -H_{IJ}. \tag{2.2}$$

A 'current' on the phase space is a function $F(x,p)$ which does not depend on the coordinate t on X_1 explicitly, that is, $\partial_t F(x,p) = 0$. Let $F(x,p)$ and $G(x,p)$ be currents on the phase space. The Poisson bracket between two currents gives us a new current:

$$\{F(x,p), G(x,p)\}_{P.B.} = \frac{\partial F}{\partial x^I}\frac{\partial G}{\partial p_I} - \frac{\partial F}{\partial p_I}\frac{\partial G}{\partial x^I} + H_{IJ}\frac{\partial F}{\partial p_I}\frac{\partial G}{\partial p_J} \equiv K(x,p), \tag{2.3}$$

from Eq. (2.2).

This structure is reformulated by an odd Poisson bracket, which is called the Schouten-Nijenhuis bracket. Let us consider the exterior algebra $\wedge^\bullet T(T^*M)$, of which sections are identified to functions on a supermanifold $T^*[1](T^*M)$, $\Gamma \wedge^\bullet T(T^*M) = C^\infty(T^*[1](T^*M))$ [3]. An odd symplectic form Ω is induced from a natural symplectic structure on a (double) cotangent bundle $T^*(T^*M)$, which is

$$\Omega = \delta \boldsymbol{x}^{\tilde{I}} \wedge \delta \boldsymbol{\xi}_{\tilde{I}} = \delta x^I \wedge \delta \xi_I + \delta p_I \wedge \delta \eta^I, \tag{2.4}$$

[3] Recent reviews about a supermanifold and a supergeometry are [17], [18] and [19].

where $\boldsymbol{x}^{\tilde{I}} = (x^I, p_I)$ is a Darboux coordinate on T^*M and $\boldsymbol{\xi}_{\tilde{I}} = (\xi_I, \eta^I)$ is an odd local coordinate of the fiber of $T^*[1](T^*M)$. Degrees are assigned to each coordinate. $\boldsymbol{x}^{\tilde{I}} = (x^I, p_I)$ has degree 0 and $\boldsymbol{\xi}_{\tilde{I}} = (\xi_I, \eta^I)$ has degree 1. The odd Poisson brackets on the canonical quantities are

$$\{\boldsymbol{x}^{\tilde{I}}, \boldsymbol{x}^{\tilde{J}}\} = 0, \quad \{\boldsymbol{\xi}_{\tilde{I}}, \boldsymbol{\xi}_{\tilde{J}}\} = 0, \quad \{\boldsymbol{x}^{\tilde{I}}, \boldsymbol{\xi}_{\tilde{J}}\} = \delta^{\tilde{I}}{}_{\tilde{J}}. \tag{2.5}$$

Now let us require a degree 2 function $\Theta \in C^\infty(T^*[1](T^*M))$ such that $\{\Theta, \Theta\} = 0$. A general solution of Θ is

$$\Theta = \frac{1}{2} f^{\tilde{I}\tilde{J}}(\boldsymbol{x}) \boldsymbol{\xi}_{\tilde{I}} \boldsymbol{\xi}_{\tilde{J}}, \tag{2.6}$$

where $f^{\tilde{I}\tilde{J}}(\boldsymbol{x})$ is skewsymmetric and $\frac{\partial f^{\tilde{I}\tilde{J}}(\boldsymbol{x})}{\partial \boldsymbol{x}^{\tilde{L}}} f^{\tilde{L}\tilde{K}}(\boldsymbol{x}) + (\tilde{I}\tilde{J}\tilde{K} \text{ cyclic}) = 0$. Since no background structure except for H_{IJ} is assumed on M, the only solution is

$$f^{\tilde{I}\tilde{J}}(\boldsymbol{x}) = \begin{pmatrix} 0 & -\delta^I{}_J \\ \delta^J{}_I & H_{IJ}(\boldsymbol{x}) \end{pmatrix}. \tag{2.7}$$

Here we set $\tilde{I} = (1, \cdots, D, D+1, \cdots, 2D)$ and $\boldsymbol{x}^{\tilde{I}} = (\boldsymbol{x}^I, \boldsymbol{x}^{D+I}) = (x^I, p_I)$. Under this settings, the original Poisson bracket is reconstructed by the derived bracket:

$$\{F(x,p), G(x,p)\}_{P.B.} = \{\{F(\boldsymbol{x}), \Theta\}, G(\boldsymbol{x})\}. \tag{2.8}$$

In fact, the Poisson brackets on the canonical conjugates are derived as

$$\{\{x^I, \Theta\}, x^J\} = 0, \quad \{\{x^I, \Theta\}, p_J\} = \delta^I{}_J, \quad \{\{p_I, \Theta\}, p_J\} = -H_{IJ}. \tag{2.9}$$

The construction here is known as a Poisson bracket from a Schouten-Nijenhuis bracket and a Poisson bivector field in the Poisson geometry [4]. A Schouten-Nijenhuis bracket $\{-,-\}$ is an odd Poisson bracket and a Poisson bivector field is Θ. The Poisson structure are associated to a structure with a Lie algebroid on $T(T^*M)$.

3 Current Algebras in Two Dimensions

In this section, the construction of current algebras by Alekseev and Strobl [1] in two dimensional spacetime is reviewed. Let us consider a two dimensional worldsheet $X_2 = S^1 \times \boldsymbol{R}$. The phase space is the cotangent bundle T^*LM of the loop space $LM = \mathrm{Map}(S^1, T^*M)$. Let

$x^I(\sigma)$ be a local coordinate and $p_I(\sigma)$ be a canonical conjugate, where σ is a local coordinate on S^1. The symplectic structure can be written as follows:

$$\omega = \int_{S^1} d\sigma \ \delta x^I \wedge \delta p_I. \tag{3.10}$$

The Poisson bracket on the canonical quantities is

$$\{x^I, x^J\}_{P.B.} = 0, \quad \{x^I, p_J\}_{P.B.} = \delta^I{}_J \delta(\sigma - \sigma'), \quad \{p_I, p_J\}_{P.B.} = 0. \tag{3.11}$$

More generally, (3.10) can be twisted by a closed 3-form H as

$$\omega = \int_{S^1} d\sigma \ \delta x^I \wedge \delta p_I + \frac{1}{2} \int_{S^1} d\sigma \ H_{IJK} \partial_\sigma x^I \delta x^J \wedge \delta x^K. \tag{3.12}$$

Then the Poisson bracket is modified to

$$\{x^I, x^J\}_{P.B.} = 0, \quad \{x^I, p_J\}_{P.B.} = \delta^I{}_J \delta(\sigma - \sigma'), \quad \{p_I, p_J\}_{P.B.} = -H_{IJK} \partial_\sigma x^K \delta(\sigma - \sigma'). \tag{3.13}$$

A generalization of a current algebra to a target space $TM \oplus T^*M$ has considered:

$$J_{0(f)}(\sigma) = f(x(\sigma)), \quad J_{1(u,\alpha)}(\sigma) = \alpha_I(x(\sigma)) \partial_\sigma x^I(\sigma) + u^I(x(\sigma)) p_I(\sigma), \tag{3.14}$$

where $f(x(\sigma))$ is a function, $\alpha_I(x) dx^I$ is a 1-form and $u = u^I(x) \partial_I$ is a vector field on the target space. $J_{0(f)}$ is a current of mass dimension zero and $J_{1(u,\alpha)}$ is one of mass dimension one. These forms contain current Kac-Moody algebras on the WZW model, currents of the Poisson sigma model and a Killing vector field with a 3-form on the target space as a special case.

The canonical commutation relations (3.13) derive commutation relations of current algebras (3.14):

$$\begin{aligned} \{J_{0(f)}(\sigma), J_{0(f')}(\sigma')\}_{P.B.} &= 0, \\ \{J_{1(u,\alpha)}(\sigma), J_{0(f')}(\sigma')\}_{P.B.} &= -u^I \frac{\partial f'}{\partial x^I}(x(\sigma)) \delta(\sigma - \sigma'), \\ \{J_{1(u,\alpha)}(\sigma), J_{1(u',\alpha')}(\sigma')\}_{P.B.} &= -J_{1([(u,\alpha),(u',\alpha')])}(\sigma) \delta(\sigma - \sigma') \\ &\quad + \langle (u,\alpha), (u',\alpha') \rangle (\sigma') \partial_\sigma \delta(\sigma - \sigma'), \end{aligned} \tag{3.15}$$

where

$$[(u,\alpha), (u',\alpha')] = ([u,u'], L_u \alpha' - L_{u'} \alpha + d(i_{u'} \alpha) + H(u, u', \cdot\,)), \tag{3.16}$$

is the Courant-Dorfman bracket on $TM \oplus T^*M$ and $\langle (u,\alpha),(u',\alpha')\rangle = i_{u'}\alpha + i_u\alpha'$ is a symmetric scalar product on $TM \oplus T^*M$ [5], [6].

From Eq. (3.15), the anomaly cancellation condition is $\langle (u,\alpha),(u',\alpha')\rangle = 0$. The current algebra closes if this condition is satisfied. This is satisfied on the Dirac structure on M. The Dirac structure is a maximally isotropic subbundle of $TM \oplus T^*M$, whose sections are closed under the Courant-Dorfman bracket.

4 QP Manifold

In this section, mathematical structures which appear in this paper, such as a QP manifold and an algebroid are prepared.

A nonnegatively graded manifold $\mathcal{M}$, called a N-manifold, is defined as a ringed space with a structure sheaf of nonnegatively graded commutative algebra over an ordinary smooth manifold M. Grading is called **degree**.

A N-manifold equipped with a graded symplectic structure (**P-structure**) Ω of degree n is called a P-manifold of degree n, $(\mathcal{M},\Omega)$. The graded Poisson bracket on $C^\infty(\mathcal{M})$ is defined from the graded symplectic structure Ω on $\mathcal{M}$ as $\{f,g\} = (-1)^{|f|+1} i_{X_f} i_{X_g}\Omega$, where a Hamiltonian vector field X_f is defined by the equation $\{f,g\} = X_f g$, for $f,g \in C^\infty(\mathcal{M})$.

Definition 4.1 *Let $(\mathcal{M},\Omega)$ be a P-manifold of degree n and Q be a differential of degree $+1$ with $Q^2=0$ on $\mathcal{M}$. Q is called a **Q-structure**. A triple $(\mathcal{M},\Omega,Q)$ is called a **QP-manifold** of degree n and its structure is called a **QP structure**, if Ω and Q are compatible, that is, $\mathcal{L}_Q\Omega = 0$* [9].

Q is also called a homological vector field. A Hamiltonian $\Theta \in C^\infty(\mathcal{M})$ of Q with respect to the graded Poisson bracket $\{-,-\}$ satisfies

$$Q = \{\Theta, -\}, \tag{4.17}$$

and has degree $n+1$. The differential condition, $Q^2 = 0$, implies that Θ is a solution of the **classical master equation**,

$$\{\Theta,\Theta\} = 0. \tag{4.18}$$

A QP manifold $(\mathcal{M},\Omega,Q)$ is also denoted by $(\mathcal{M},\Omega,\Theta)$.

$T^*[1](T^*M)$ in section 2 is a graded manifold of degree 1. An odd symplectic form Ω is a P-structure of degree 1 and a Poisson bivector field Θ is a Q-structure. Thus a current algebra in one dimension has a structure of a QP-manifold of degree 1.

Definition 4.2 *A vector bundle $E = (E, \rho, [-,-])$ is called* ***an algebroid*** *if there is a bracket product $[e_1, e_2]$, where $e_1, e_2 \in \Gamma E$, and a bundle map $\rho : E \to TM$ which is called an anchor map, satisfying the conditions below:*

$$\rho[e_1, e_2] = [\rho(e_1), \rho(e_2)], \tag{4.19}$$

$$[e_1, fe_2] = f[e_1, e_2] + \rho(e_1)(f)e_2, \tag{4.20}$$

where the bracket $[\rho(e_1), \rho(e_2)]$ is the usual Lie bracket on ΓTM.

A Loday algebroid is an algebroid version of a Loday(Leibniz) algebra [20], [21].

Definition 4.3 *An algebroid $E = (E, \rho, [-,-])$ is called* ***a Loday algebroid*** *if there is a bracket product $[e_1, e_2]$ satisfying the Leibniz identity:*

$$[e_1, [e_2, e_3]] \;=\; [[e_1, e_2], e_3] + [e_2, [e_1, e_3]], \tag{4.21}$$

where $e_1, e_2, e_3 \in \Gamma E$, A Loday algebroid is also called a Leibniz algebroid.

Correspondence of a Loday algebroid with a homological vector field on a supermanifold is discussed in [22]. The following theorem has appeared in [17].

Theorem 4.4 *Let $n > 1$. Functions of degree $n-1$ on a QP manifold can be identifies as sections of a vector bundle E. The QP-structure induces a Loday algebroid structure on E.*

Let $\boldsymbol{x}$ be an element of degree 0 and $\boldsymbol{e}^{(n-1)}$ be the element of degree $n-1$. If we define

$$[e_1, e_2] \;=\; \{\{\boldsymbol{e}_1^{(n-1)}, \Theta\}, \boldsymbol{e}_2^{(n-1)}\}, \tag{4.22}$$

$$\rho(e_1)F(x) \;=\; \{\{\boldsymbol{e}^{(n-1)}, \Theta\}, F(\boldsymbol{x})\}, \tag{4.23}$$

where $[-,-]$ and ρ satisfy the relations of a Loday algebroid (4.19), (4.20) and (4.21).

Examples of a QP manifold of degree n are listed.

Example 4.5 Let $\mathfrak{g}$ be a Lie algebra with a Lie bracket $[-,-]$. Then $T^*[n]\mathfrak{g}[1]$ is a QP manifold of degree n. A natural P-structure Ω is induced from the canonical symplectic structure on $T^*\mathfrak{g}$, constructed from the canonical pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$, $\langle -,-\rangle$. Define $\Theta = \frac{1}{2}\langle \boldsymbol{p}, [\boldsymbol{q}, \boldsymbol{q}]\rangle$, where $\boldsymbol{q} \in \mathfrak{g}[1]$ and $\boldsymbol{p} \in \mathfrak{g}^*[n]$. Since $\{\Theta, \Theta\} = 0$ from a Lie algebra structure, Θ defines a Q-structure. If we take a structure constant $f^A{}_{BC}$ of Lie algebra, $\Theta = \frac{1}{2} f^A{}_{BC} \boldsymbol{p}_A \boldsymbol{q}^B \boldsymbol{q}^C$.

Example 4.6 Let $n = 1$. Then $\mathcal{M}$ is canonically $\mathcal{M} = T^*[1]M$ and a Poisson bracket $\{-,-\}$ in the P-structure is a Schouten-Nijenhuis bracket. A Q-structure Θ has degree 2 and $Q^2 = 0$ is that Θ is a Poisson bivector field. Thus a QP manifold of degree 1 is a Poisson manifold on M.

Example 4.7 Let $n = 2$. A P-structure Ω is an even form of degree 2. A Q-structure Θ has degree 3 and $Q^2 = 0$ defines a Courant algebroid structure on a vector bundle E. A QP manifold of degree 2 is a Courant algebroid [7].

The Dirac structure L is a maximally isotropic subbundle of the Courant algebroid E, whose sections are closed under the Courant-Dorfman bracket. The symmetric scalar product $\langle -,-\rangle$ corresponds to the Q-structure $\{-,-\}$ of a QP manifold construction of the Courant algebroid. If we identify functions on a QP manifold to the sections of a vector bundle E, the sections of the Dirac structure ΓL are commutative under the P-structure $\{-,-\}$ and closed under the derived bracket $\{\{-,\Theta\},-\}$.

Example 4.8 Let $n = 3$. One of examples of a N-manifold is $\mathcal{M} := T^*[3]E[1]$. A P-structure Ω is an odd form of degree 3. A Q-structure $Q^2 = 0$ defines a Lie algebroid up to homotopy (the splittable H-twisted Lie algebroid) on E [15]. A general nonsplittable algebroid is the H-twisted Lie algebroid [16].

Example 4.9 Let E be a vector bundle on M and $\mathcal{M} = T^*[n]E[1]$. If a QP structure is defined on $\mathcal{M} = T^*[n]E[1]$ and $n \geq 4$, E becomes a Lie algebroid and $\Gamma E \oplus \wedge^{n-1}E^*$ is a subalgebroid. A Lie algebroid is a Loday algebroid which bracket $[-,-]$ is skewsymmetric. A QP structure induces the Courant-Dorfman bracket on the subalgebroid $E \oplus \wedge^{n-1}E^*$ by the derived bracket $[-,-] = \{\{-,\Theta\},-\}$, which has the following form,

$$[u+\alpha, v+\beta] = [u,v] + L_u\beta - i_v d\alpha + H(u,v), \tag{4.24}$$

where $u, v \in \Gamma E$, $\alpha, \beta \in \Gamma \wedge^{n-1} E^*$ and H is a closed $(n+1)$-form on E. We refer the reader to [24], [25] and [26] for detailed studies of the bracket of this type. QP descriptions of higher Courant-Dorfman brackets are discussed in [27].

5 QP Structures of Current Algebras in Two Dimensions

The construction of the mechanics by the odd Poisson (Schouten-Nijenhuis) bracket in section 2 is generalized to current algebras in two dimensions. Let us consider the super extension of the space direction of the worldsheet S^1, $T[1]S^1$, which has a local coordinate (σ, θ). Moreover we consider the graded extension of degree 2 of the target space, $\mathcal{M} = T^*[2](T^*[1]M)$ and the space of smooth map from $T[1]S^1$ to $T^*[2](T^*[1]M)$, which is denoted by $\mathrm{Map}(T[1]S^1, T^*[2](T^*[1]M))$. A QP structure of degree 2 is introduced on $T^*[2](T^*[1]M)$. Let Ω be a P-structure and Θ be a Q-structure on $T^*[2](T^*[1]M)$

$x^I(\sigma)$ is extended to a superfield $\boldsymbol{x}^I(\sigma, \theta) = x^I(\sigma) + \theta x^{(1)I}(\sigma)$, which is a smooth map from $T[1]S^1$ to M. The canonical conjugate p_I is extended to an odd superfield of degree 1, $\boldsymbol{p}_I(\sigma, \theta) = p_I^{(0)}(\sigma) + \theta p_I(\sigma)$, which is a section of $T^*[1]S^1 \otimes \boldsymbol{x}^*(T^*[1]M)$, where $p_I^{(0)}$ is an odd auxiliary field of degree 1, Moreover Let us introduce 'canonical conjugates' of $\boldsymbol{p}_I$ and $\boldsymbol{x}^I$, respectively. A superfield $\boldsymbol{\eta}^I(\sigma, \theta) = \eta^{(0)I}(\sigma) + \theta \eta^{(1)I}(\sigma)$ of degree 1 is a section of $T^*[1]S^1 \otimes \boldsymbol{x}^*(T^*[1]M)$ and $\boldsymbol{\xi}_I(\sigma, \theta) = \xi_I^{(0)}(\sigma) + \theta \xi_I^{(1)}(\sigma)$ of degree 2 is a section of $T^*[1]S^1 \otimes \boldsymbol{x}^*(T^*[2]M)$.

The Courant algebroid structure on $TM \oplus T^*M$ is mapped by the canonical shifting and embedding called the minimal symplectic realization $j : TM \oplus T^*M \longrightarrow T^*[2]T^*[1]M$ such that $j : (x^I, \frac{\partial}{\partial x^I}, 0, dx^I) \longmapsto (x^I, p_I, \xi_I, \eta^I)$. $\frac{\partial}{\partial x^I} \longmapsto p_I$ is defined by a natural pairing of TM and T^*M, $\langle u^I \frac{\partial}{\partial x^I}, p_I dx^I \rangle \mapsto u^I p_I$. j induces a map $\widehat{j} : (x^I, p_I, 0, dx^I) \longmapsto (\boldsymbol{x}^I, \boldsymbol{p}_I, \boldsymbol{\xi}_I, \boldsymbol{\eta}^I)$ for superfields.

A graded symplectic form $\boldsymbol{\Omega}$ is defined from Ω as

$$\boldsymbol{\Omega} = \int_{T[1]S^1} d\sigma d\theta \ \Phi^* \Omega = \int_{T[1]S^1} d\sigma d\theta \ (\delta \boldsymbol{x}^I \wedge \delta \boldsymbol{\xi}_I + \delta \boldsymbol{p}_I \wedge \delta \boldsymbol{\eta}^I), \tag{5.25}$$

where $\Phi \in \mathrm{Map}(T[1]S^1, T^*[2](T^*[1]M))$. Then the commutation relations are obtained as

$$\{\boldsymbol{p}_J(\sigma,\theta), \boldsymbol{\eta}^I(\sigma',\theta')\} = \{\boldsymbol{\eta}^I(\sigma,\theta), \boldsymbol{p}_J(\sigma',\theta')\} = \delta^I{}_J \delta(\sigma-\sigma')\delta(\theta-\theta'),$$
$$\{\boldsymbol{x}^I(\sigma,\theta), \boldsymbol{\xi}_J(\sigma',\theta)\} = -\{\boldsymbol{\xi}_J(\sigma,\theta), \boldsymbol{x}^I(\sigma',\theta)\} = \delta^I{}_J \delta(\sigma-\sigma')\delta(\theta-\theta'), \tag{5.26}$$

where $\delta(\theta-\theta')$ is defined by $\theta+\theta'$.

Next a Q-structure Hamiltonian functional $\boldsymbol{\Theta}$ on $\mathrm{Map}(T[1]S^1, \mathcal{M})$ is constructed. $\boldsymbol{\Theta}$ is induced from the Q-structure Hamiltonian function Θ on $\mathcal{M}$. $\boldsymbol{\Theta}$ is defined as follows:

$$\boldsymbol{\Theta} = \int_{T[1]S^1} d\sigma d\theta \ \Phi^*\Theta. \tag{5.27}$$

Since the integration shifts degree by 1, $(\mathrm{Map}(T[1]S^1, \mathcal{M}), \boldsymbol{\Omega}, \boldsymbol{\Theta})$ is a QP manifold of degree 1. Therefore $\boldsymbol{\Omega}, \boldsymbol{\Theta}$ define a Poisson structure.

Since $\mathcal{M} = T^*[2](T^*[1]M)$ is a QP-manifold of degree 2, a Hamiltonian function Θ is of degree 3 and defines the Courant-Dorfman bracket from Ex. 4.7 by the derived bracket:

$$[-,-] = \{\{-,\Theta\},-\}. \tag{5.28}$$

Since we do not consider any extra structure except for a closed 3-form H in the original two dimensional system, a Hamiltonian function is canonically taken as

$$\Phi^*\Theta = \boldsymbol{\eta}^I \boldsymbol{\xi}_I + \frac{1}{3!} H_{IJK}(\boldsymbol{x}) \boldsymbol{\eta}^I \boldsymbol{\eta}^J \boldsymbol{\eta}^K. \tag{5.29}$$

$\{\Theta,\Theta\} = 0$ if and only if H is a closed form. Indeed the derived brackets of graded fields are obtained as follows:

$$\{\{\boldsymbol{x}^I(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{x}^J(\sigma',\theta')\} = 0,$$
$$\{\{\boldsymbol{x}^I(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{p}_J(\sigma',\theta')\} = \delta^I{}_J \delta(\sigma-\sigma')\delta(\theta-\theta'),$$
$$\{\{\boldsymbol{p}_I(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{p}_J(\sigma',\theta')\} = -H_{IJK}(\boldsymbol{x}(\sigma,\theta))\boldsymbol{\eta}^K(\sigma,\theta)\delta(\sigma-\sigma')\delta(\theta-\theta'). \tag{5.30}$$

The pullback of the embedding map $\widehat{j}_*$ induce the Poisson bracket on the original fields. This corresponds to taking a projection to a submanifold by making the projection of auxiliary fields,

$$x^{(1)I}(\sigma) = p^{(0)}_I(\sigma) = \eta^{(0)I} = 0, \qquad \eta^{(1)I} = \partial x^I, \tag{5.31}$$

after the calculations of the graded Poisson bracket. The Poisson brackets (3.13) of canonical quantities are derived as

$$\begin{aligned}
\{x^I(\sigma), x^J(\sigma')\}_{P.B.} &= \widehat{j}^*\{\{\boldsymbol{x}^I(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{x}^J(\sigma',\theta')\}, \\
\{x^I(\sigma), p_J(\sigma')\}_{P.B.} &= \widehat{j}^*\{\{\boldsymbol{x}^I(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{p}_J(\sigma',\theta')\}, \\
\{p_I(\sigma), p_J(\sigma')\}_{P.B.} &= \widehat{j}^*\{\{\boldsymbol{p}_I(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{p}_J(\sigma',\theta')\}. \qquad (5.32)
\end{aligned}$$

Generalized currents are converted to super functions of degree of zero and degree of one on $C^\infty(\mathcal{M})$ by $\widehat{j}_* : J \longmapsto \boldsymbol{J}$, respectively:

$$\boldsymbol{J}_{0(f)}(\sigma,\theta) = f(\boldsymbol{x}), \quad \text{and} \quad \boldsymbol{J}_{1(u,\alpha)}(\sigma,\theta) = \alpha_I(\boldsymbol{x})\boldsymbol{\eta}^I + u^I(\boldsymbol{x})\boldsymbol{p}_I. \qquad (5.33)$$

The original currents are recovered by the pullback of $\widehat{j}$. The derived brackets of these current functions are directly calculated from (5.26) as follows:

$$\begin{aligned}
&\{\{\boldsymbol{J}_{0(f)}(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{J}_{0(f')}(\sigma',\theta')\} = 0, \\
&\{\{\boldsymbol{J}_{1(u,\alpha)}(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{J}_{0(f')}(\sigma',\theta')\} = -u'^I \frac{\partial f}{\partial \boldsymbol{x}^I}\delta(\sigma-\sigma')\delta(\theta-\theta'), \\
&\{\{\boldsymbol{J}_{1(u,\alpha)}(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{J}_{1(u',\alpha')}(\sigma',\theta')\} = -\boldsymbol{J}_{1([(u,\alpha),(u',\alpha')])}(\sigma,\theta)\delta(\sigma-\sigma')\delta(\theta-\theta'), \quad (5.34)
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{J}_{1([(u,\alpha),(u',\alpha')])}(\sigma,\theta) = \Bigg[&\left(u^J \frac{\partial u'^I}{\partial \boldsymbol{x}^J} - u'^J \frac{\partial u^I}{\partial \boldsymbol{x}^J} \right) \boldsymbol{p}_I \\
&+ \left(u^J \frac{\partial \alpha'_I}{\partial \boldsymbol{x}^J} - u'^J \frac{\partial \alpha_I}{\partial \boldsymbol{x}^J} + u'^J \frac{\partial \alpha_J}{\partial \boldsymbol{x}^I} + \alpha'_J \frac{\partial u^J}{\partial \boldsymbol{x}^I} + H_{JKI} u^J u'^K \right) \boldsymbol{\eta}^I \Bigg]. \quad (5.35)
\end{aligned}$$

Here Eq. (5.28) is used to derive the Courant bracket. Eq. (5.34) has the same relations as the current algebras (3.15) except for the anomaly terms.

The anomaly terms, which are proportional to the differentials of the delta function, are obtained by the graded Poisson brackets of currents as

$$\begin{aligned}
&\{\boldsymbol{J}_{0(f)}(\sigma,\theta), \boldsymbol{J}_{0(f')}(\sigma',\theta')\} = 0, \quad \{\boldsymbol{J}_{0(f)}(\sigma,\theta), \boldsymbol{J}_{1(u',\alpha')}(\sigma',\theta')\} = 0, \\
&\{\boldsymbol{J}_{1(u,\alpha)}(\sigma,\theta), \boldsymbol{J}_{1(u',\alpha')}(\sigma',\theta')\} = (\alpha_I u'^I + \alpha'_I u^I)\delta(\sigma-\sigma')\delta(\theta-\theta'). \qquad (5.36)
\end{aligned}$$

The coefficients of the delta functions in the right hand sides are the same as the coefficients of $\partial_\sigma\delta(\sigma-\sigma')$ in anomaly terms in the current algebra (3.15). Results are summarized to the following theorem:

Theorem 5.1 *A current algebra* (3.15) *in two dimensions has a realization in a QP manifold of degree* 2 *on* $T^*[2]T^*[1]M$ *induced from the minimal symplectic realization* $j : TM \oplus T^*M \longrightarrow T^*[2]T^*[1]M$*. The algebraic structure is calculated by the derived bracket of current super functions. The anomaly cancellation condition is equivalent to the condition that currents super functions are commutative under the Q-structure.*

6 Generalized Current Algebras in Three Dimensions

Current algebras in the previous sections are generalized to three dimensions. It is necessary to extend current algebras in three dimensions in order to contain currents in a large class of field theories such as the Chern-Simons theory with matters or the Courant sigma model.

6.1 Chern-Simons Theory with Matter

The action of complex scalar fields coupled with the Chern-Simons theory in three dimensions is

$$\begin{aligned} S &= \int_{X_3} d^3\sigma \Big[k_{IJ}(\partial_\mu x^I + f^I{}_{KL} q_\mu^K x^L)(\partial^\mu x^{*J} + f^J{}_{MN} q^{\mu M} x^{*N}) \\ &\qquad + V(k_{IJ} x^{*I} x^J) + \frac{k_{IJ}}{2} \varepsilon^{\mu\nu\rho} q_\mu^I \partial_\nu q_\rho^J + \frac{1}{3!} f_{IJK} \varepsilon^{\mu\nu\rho} q_\mu^I q_\nu^J q_\rho^K \Big]. \end{aligned} \tag{6.37}$$

where $X_3 = \Sigma \times \boldsymbol{R}$ is a manifold in three dimensions, x^I is a complex scalar field and q_μ^K is a gauge field. f_{IJK} is a structure constant of a Lie algebra, k_{IJ} is a metric on a Lie algebra Recently, this action is used to describe multiple M2-branes [28] [29] [30] [31]. The action (6.37) can be written in terms of differential form as follows:

$$S = \int_{X_3} k_{IJ}(dx^I + [q,x]^I) \wedge *(dx^{*J} + [q,x^*]^J) + *V(k_{IJ} x^{*I} x^J) + \frac{k_{IJ}}{2} q^I \wedge dq^J + \frac{1}{3!} f_{IJK} q^I \wedge q^J \wedge q^K, \tag{6.38}$$

where

$$[q,x]^I = f^I{}_{JK} q^J x^K, \tag{6.39}$$

and $x^I(\sigma)$ and $q^I(\sigma) = d\sigma^\mu q^I_\mu(\sigma)$ are a 0-form and a 1-form, respectively. $*$ represents the Hodge star. The canonical momenta are

$$\begin{aligned} p_I &= \frac{\delta S}{\delta(\partial_0 x^I)} = k_{IJ}(\partial_0 x^J + f^J{}_{KL} q_0^K x^L), \\ p_I^* &= \frac{\delta S}{\delta(\partial_0 x^{*I})} = k_{IJ}(\partial_0 x^{*J} + f^J{}_{KL} q_0^K x^{*L}), \\ \pi_I &= \frac{\delta S}{\delta(\partial_0 q_1^I)} = k_{IJ} q_2^J. \end{aligned} \qquad (6.40)$$

The symplectic structure is defined as

$$\omega = \int_\Sigma d^2\sigma \left(\delta x^I \wedge \delta p_I + \delta x^{*I} \wedge \delta p_I^* + k_{IJ} \delta q_1^I \wedge \delta q_2^J\right). \qquad (6.41)$$

The Hamiltonian of the system is following:

$$\mathcal{H} = \mathcal{H}_0 + q_0^I J_{1I}, \qquad (6.42)$$

where J_{1I} is a current of a gauge symmetry. Concrete expressions are

$$\mathcal{H}_0 = k^{IJ} p_I^* p_J - k_{IJ} \sum_{i=1}^{2} (\partial_i x^I + f^I{}_{KL} q_i^K x^L)(\partial_i x^{*J} + f^J{}_{MN} q_i^M x^{*N}) - V(x), \qquad (6.43)$$

$$J_{1I} = k_{IJ}(\partial_2 q_1^J - \partial_1 q_2^J) - f_{IJK} q_1^J q_2^K - f_{IJ}{}^K (x^J p_K + x^{*J} p_K^*). \qquad (6.44)$$

The Poisson bracket for gauge currents is as follows:

$$\{J_{1I}(\sigma), J_{1J}(\sigma')\}_{P.B.} = -f_{IJ}{}^K J_{1K}(\sigma)\delta(\sigma - \sigma') \approx 0, \qquad (6.45)$$

where we use the Jacobi identity:

$$k^{KN}(f_{KIL} f_{JMN} + f_{KLJ} f_{IMN} + f_{KIJ} f_{MLN}) = 0. \qquad (6.46)$$

We can easily confirm that J_{1I} commutes with the Hamiltonian,

$$\{J_{1I}(\sigma), \mathcal{H}(\sigma')\}_{P.B.} = -q_0^J f_{IJ}^K J_{1K} \delta(\sigma - \sigma') \approx 0. \qquad (6.47)$$

6.2 Courant Sigma Model

The Courant sigma model has been introduced in [10] and formulated by the AKSZ formalism in [11], which is a topological sigma model in the $1+2$-dimensional worldvolume.

This is a sigma model on a three dimensional manifold X_3 to a vector bundle $E \longrightarrow M$. This has the following action:

$$S = \int_{X_3} p_I \wedge dx^I + \frac{k_{AB}}{2} q^A \wedge dq^B - f_1{}^I{}_A(x) q^A \wedge p_I + \frac{1}{3!} f_{2ABC}(x) q^A \wedge q^B \wedge q^C, \qquad (6.48)$$

where x^I, q^A and p_I are 0-form, 1-form and 2-form on the base manifold X_3, respectively. $I, J, \cdots$ are indices of M, and $A, B, \cdots$ are indices of the fiber of E, and k_{AB} is a fiber metric on E. By the construction, note that k_{AB} is symmetric for A, B and $f_{2ABC}(x)$ is skewsymmetric for A, B, C. More precisely, x^I is a smooth map from X_3 to M, $q^A \in \Gamma(T^*X_3 \otimes x^*(E))$ and $p_I \in \Gamma(\wedge^2 T^*X_3 \otimes x^*(T^*M))$.

For consistency of the action, the structure functions $f_1{}^I{}_A(x)$, $f_{2ABC}(x)$ must have the following conditions (see [10]):

$$\begin{aligned}
&(1) \quad k^{AB} f_{1A}^I f_{1B}^J = 0, \\
&(2) \quad f_{1A}^J \frac{\partial f_{1B}^I}{\partial x^J} - f_{1B}^J \frac{\partial f_{1A}^I}{\partial x^J} + k^{CD} f_{1C}^I f_{2DBA} = 0, \\
&(3) \quad f_{1D}^I \frac{\partial f_{2ABC}}{\partial x^I} - f_{1A}^I \frac{\partial f_{2BCD}}{\partial x^I} + f_{1B}^I \frac{\partial f_{2CDA}}{\partial x^I} - f_{1C}^I \frac{\partial f_{2DAB}}{\partial x^I} \\
&\qquad\qquad + k^{EF}(f_{2EAB} f_{2CDF} + f_{2EAC} f_{2DBF} + f_{2EAD} f_{2BCF}) = 0. \qquad (6.49)
\end{aligned}$$

(6.49) is equivalent to the condition that a vector bundle E is the Courant algebroid [10], [11]. If we take $f_{1A}^I(x) = 0$ and $f_{2ABC}(x) = f_{2ABC}$ =constant, the Courant algebroid reduces to a Lie algebra and the action (6.48) reduces to the Chern-Simons gauge theory plus a BF theory. This model contains a lot of known models, such as the Chern-Simons gauge theory and the Rozansky-Witten theory [23].

We set $X_3 = \Sigma_2 \times \boldsymbol{R}$, and $q^A = q_\mu^A d\sigma^\mu$ and $p_I = \frac{1}{2} p_{I\mu\nu} d\sigma^\mu d\sigma^\nu$. Since the canonical momenta yield

$$\pi_I = \frac{\delta S}{\delta(\partial_0 x^I)} = p_{I12} = -p_{I21}, \quad \pi_A = \frac{\delta S}{\delta(\partial_0 q_1^A)} = k_{BA} q_2^B, \qquad (6.50)$$

the symplectic structure is written as

$$\omega = \int_{\Sigma_2} d^2\sigma \left(\delta x^I \wedge \delta p_{I12} + k_{AB} \delta q_1^A \wedge \delta q_2^B \right). \qquad (6.51)$$

The Hamiltonian of the system is following:

$$\begin{aligned}
\mathcal{H} &= \int_{\Sigma_2} d^2\sigma \left(\pi_I \partial_0 x^I + \pi_A \partial_0 q_1^A - \mathcal{L}\right) \\
&= \int_{\Sigma_2} d^2\sigma \Big(- p_{I01}(\partial_2 x^I - f_{1A}^I q_2^A) + p_{I02}(\partial_1 x^I - f_{1A}^I q_1^A) \\
&\qquad\qquad -q_0^A \left(k_{AB}(\partial_1 q_2^B - \partial_2 q_1^B) - f_{1A}^I p_{I12} + f_{2ABC} q_1^B q_2^C\right) \Big). \qquad (6.52)
\end{aligned}$$

Since the fields p_{I01}, p_{I02}, q_0^A are not dynamical variables, the following quantities are referred as currents,

$$\begin{aligned}
J_{1i}^I(\sigma) &= \partial_i x^I(\sigma) - f_{1A}^I(x(\sigma)) q_i^A(\sigma), \\
J_{2A}(\sigma) &= k_{AB}(\partial_1 q_2^B(\sigma) - \partial_2 q_1^B(\sigma)) - f_{1A}^I(x(\sigma)) p_{I12}(\sigma) + f_{2ABC}(x(\sigma)) q_1^B(\sigma) q_2^C(\sigma), \qquad (6.53)
\end{aligned}$$

where $i = 1, 2$. In fact, they are constraints $J_{1i}^I(\sigma) \approx 0$ and $J_{2A}(\sigma) \approx 0$. The Hamiltonian is expressed in term of these currents as follows:

$$\mathcal{H} = \int_{\Sigma_2} d^2\sigma (-p_{I01} J_{12}^I + p_{I02} J_{11}^I - q_0^A J_{2A}). \qquad (6.54)$$

For consistency of the system, the Poisson brackets of constraints must be closed on the constraint subspace. The current algebra takes the following commutation relations:

$$\begin{aligned}
&\{J_{1i}^I(\sigma), J_{1j}^J(\sigma')\}_{P.B.} = 0, \\
&\{J_{2A}(\sigma), J_{1i}^I(\sigma')\}_{P.B.} = \frac{\partial f_{1A}^I}{\partial x^J}(x(\sigma)) J_{1i}^J(\sigma) \delta^2(\sigma - \sigma'), \\
&\{J_{2A}(\sigma), J_{2B}(\sigma')\}_{P.B.} = \Big(\frac{\partial f_{2ABC}}{\partial x^I}(x(\sigma))(J_{11}^I(\sigma) q_2^C(\sigma) - J_{12}^I(\sigma) q_1^C(\sigma)) \\
&\qquad\qquad + f_{2ABC}(x(\sigma)) k^{CD} J_{2D}(\sigma) \Big) \delta^2(\sigma - \sigma'), \qquad (6.55)
\end{aligned}$$

with Eq. (6.49).

6.3 Generalized Current Algebras in Three Dimensions

We generalize a current algebra in three dimensional worldvolume as a generalization of Alekseev and Strobl, and in order to contain the currents of the matters coupled with the Chern-Simons theory or the Courant sigma model in the previous subsections.

Let us take the worldvolume $X_3 = \Sigma_2 \times \boldsymbol{R}$ and a target vector bundle E. Since the Courant sigma model is generalized by introducing a closed 4-form [32], the symplectic structure (6.51)

can be generalized by introducing a closed 4-form $H = \frac{1}{4!}H_{IJKL}(x)dx^I dx^J dx^K dx^L$ on a target space M, which is

$$\omega = \int_{\Sigma_2} d^2\sigma(\delta x^I \wedge \delta p_{I12} + k^{AB}\delta q_1^A \wedge \delta q_2^B) + \frac{1}{4}\int_{\Sigma_2} d^2\sigma H_{IJKL}\epsilon^{ij}\partial_i x^I \partial_j x^J \delta x^K \wedge \delta x^L, \quad (6.56)$$

where $i, j = 1, 2$ and $\epsilon^{12} = 1, \epsilon^{21} = -1$ (and others are 0). Then the Poisson brackets of canonical quantities are

$$\begin{aligned} &\{x^I(\sigma), p_{Jij}(\sigma')\}_{P.B.} = \epsilon_{ij}\delta^I_J\delta^2(\sigma-\sigma'), \quad \{q_i^A(\sigma), q_j^B(\sigma')\}_{P.B.} = \epsilon_{ij}k^{AB}\delta^2(\sigma-\sigma'), \\ &\{p_{Iij}(\sigma), p_{Jkl}(\sigma')\}_{P.B.} = -\frac{1}{2}\epsilon_{ij}\epsilon_{kl}H_{IJKL}(x)\epsilon^{mn}\partial_m x^K \partial_n x^L \delta^2(\sigma-\sigma'). \end{aligned} \quad (6.57)$$

The mass dimensions of each quantities are chosen so that ω is dimensionless [2]. Since $\dim[\sigma] = -1$ and $\dim[\partial] = 1$, the mass dimensions of the canonical conjugates are taken as $\dim[x^I] = 0$, $\dim[q^A] = 1$ and $\dim[p_I] = 2$.

If it is assumed that each current has the homogeneous mass dimension, the following three currents are most general forms of mass dimension zero, one and two, respectively:

$$\begin{aligned} &J_{0(f)}(\sigma) = f(x(\sigma)), \\ &J_{1i(\alpha,u)}(\sigma) = \alpha_I(x(\sigma))\partial_i x^I(\sigma) + u_A(x(\sigma))q_i^A(\sigma), \\ &J_{2ij(G,K,F,B,E)}(\sigma) = \epsilon_{ij}\epsilon^{kl}\left(\frac{1}{2}G^I(x(\sigma))p_{Ikl}(\sigma) + K_A(x(\sigma))\partial_k q_l^A(\sigma) + \frac{1}{2}F_{AB}(x(\sigma))q_k^A(\sigma)q_l^B(\sigma)\right. \\ &\qquad \left. + \frac{1}{2}B_{IJ}(x(\sigma))\partial_k x^I(\sigma)\partial_l x^J(\sigma) + E_{AI}(x(\sigma))\partial_k x^I(\sigma) q_l^A(\sigma)\right). \end{aligned} \quad (6.58)$$

If the canonical quantities are rewritten by differential forms $q^A = q_i^A d\sigma^i$, $p_I = \frac{1}{2}p_{Iij}d\sigma^i \wedge d\sigma^j$ on the space direction Σ, the canonical commutation relations are

$$\begin{aligned} &\{x^I(\sigma), p_J(\sigma')\} = \delta^I_J\delta^2(\sigma-\sigma'), \quad \{q^A(\sigma), q^B(\sigma')\} = -k^{AB}\delta^2(\sigma-\sigma'), \\ &\{p_I(\sigma), p_J(\sigma')\} = -\frac{1}{2}H_{IJKL}dx^K \wedge dx^L \delta^2(\sigma-\sigma') \end{aligned} \quad (6.59)$$

Generalized currents in differential form are written by

$$J_{0(f)}(\sigma) = f, \quad J_{1(\alpha,u)}(\sigma) = \alpha + u, \quad J_{2(G,K,F,B,E)}(\sigma) = G + K + F + B + E, \quad (6.60)$$

respectively. Here α, u, G, K, F, H, E are defined as forms:

$$\begin{aligned} &\alpha = \alpha_I(x(\sigma))dx^I, \quad u = u_A(x(\sigma))q^A, \quad G = G^I(x(\sigma))p_I(\sigma), \quad K = K_A dq^A, \\ &F = \frac{1}{2}F_{AB}(x(\sigma))q^A(\sigma) \wedge q^B(\sigma), \quad B = \frac{1}{2}B_{IJ}(x(\sigma))dx^I(\sigma) \wedge dx^J(\sigma), \\ &E = E_{AI}(x(\sigma))dx^I(\sigma) \wedge q^A(\sigma). \end{aligned} \quad (6.61)$$

The Poisson brackets of currents are directly calculated by using (6.59) as the following forms:

$$
\begin{aligned}
&\{J_{0(f)}(\sigma), J_{0(f')}(\sigma')\} = 0, \\
&\{J_{1(\alpha,u)}(\sigma), J_{0(f')}(\sigma')\} = 0, \\
&\{J_{2(G,K,F,B,E)}(\sigma), J_{0(f')}(\sigma')\} = -i_G df' \delta^2(\sigma - \sigma'), \\
&\{J_{1(\alpha,u)}(\sigma), J_{1(\alpha',u')}(\sigma')\} = -\langle u, u' \rangle \delta^2(\sigma - \sigma'), \\
&\{J_{2(G,K,F,B,E)}(\sigma), J_{1(\alpha',u')}(\sigma')\} \\
&\qquad\qquad = -J_{1(\bar{\alpha},\bar{u})} \delta^2(\sigma - \sigma') + (i_G \alpha' - \langle u', K \rangle) d\delta^2(\sigma - \sigma'), \\
&\{J_{2(G,K,F,B,E)}(\sigma), J_{2(G',K',F',B',E')}(\sigma')\} \\
&\quad = -J_{2(\bar{G},\bar{K},\bar{F},\bar{B},\bar{E})} \delta^2(\sigma - \sigma') \\
&\quad -(i_G(E' + B') + i'_G(E + B) + \langle E' + F', K \rangle + \langle E + F, K' \rangle) \wedge d\delta^2(\sigma - \sigma'),
\end{aligned} \tag{6.62}
$$

and

$$
\begin{aligned}
&\bar{\alpha} = (i_G d + d i_G)\alpha' + \langle E - dK, u' \rangle, \quad \bar{u} = i_G du' + \langle F, u' \rangle, \\
&\bar{G} = [G, G'], \\
&\bar{K} = i_G dK' - i_{G'} dK + i_{G'} E + \langle F, K' \rangle, \\
&\bar{F} = i_G dF' - i_{G'} dF + \langle F, F' \rangle, \\
&\bar{B} = (d i_G + i_G d)B' - i_{G'} dB + \langle E, E' \rangle + \langle K', dE \rangle - \langle dK, E' \rangle + i_{G'} i_G H, \\
&\bar{E} = (d i_G + i_G d)E' - i_{G'} dE + \langle E, F' \rangle - \langle E', F \rangle + \langle dF, K' \rangle - \langle dK, F' \rangle,
\end{aligned} \tag{6.63}
$$

where all the terms are evaluated by σ'. Here $[-,-]$ is a Lie bracket on TM, i_G is an interior product with respect to a vector field G and $\langle -,- \rangle$ is the graded bilinear form on the fiber of E with respect to the metric k^{AB}. For example, $\langle q^A, q^B \rangle = k^{AB}$, $\langle q^A \wedge q^B, q^C \rangle = q^A k^{BC} - q^B k^{AC}$, etc. Component expressions of the equations (6.62) and (6.63) appears in the Appendix.

6.4 Current Algebras and QP structures of degree 3

We point out that Current algebras in the previous subsection are constructed from a QP structure of degree 3 on $\mathcal{M} = T^*[3]T^*[2]E[1]$ in this subsection.

Let us extend the space direction of the worldvolume to the supermanifold $T[1]\Sigma_2$ with a local coordinate (σ^i, θ^i). Let $(T^*[3]T^*[2]E[1], \Omega, \Theta)$ be a QP manifold of degree 3. A

target space T^*E has a natural symplectic realization $j : T^*E \longrightarrow T^*[3]T^*[2]E[1]$, where $j : (x^I, q^A, \frac{\partial}{\partial x^I}, 0, dq^A, dx^I) \longmapsto (x^I, q^A, p_I, \xi_I, \eta^A, \chi^I)$. This map and a natural pairing of TM and T^*M induce an embedding map of mapping space of superfields $\widehat{j} : (x^I, q^A, p_I, 0, dq^A, dx^I) \longmapsto (\boldsymbol{x}^I, \boldsymbol{q}^A, \boldsymbol{p}_I, \boldsymbol{\xi}_I, \boldsymbol{\eta}^A, \boldsymbol{\chi}^I)$. Let us consider the space of smooth map from $T[1]\Sigma_2$ to $T^*[3](T^*[2]E[1])$, which is denoted by $\mathrm{Map}(T[1]\Sigma_2, T^*[3](T^*[2]E[1]))$.

Let us consider a local coordinate expression. Let $\boldsymbol{x}^I(\sigma,\theta) = x^I(\sigma) + \theta^i x_i^{(1)I}(\sigma) + \frac{1}{2}\theta^i\theta^j x_{ij}^{(2)I}(\sigma)$ be a smooth map from $T[1]\Sigma_2$ to M. $\boldsymbol{q}^A(\sigma,\theta) \in \Gamma(T^*[1]\Sigma_2 \otimes \boldsymbol{x}^*(E[1]))$ is a superfield of degree 1, $\boldsymbol{q}^A(\sigma,\theta) = q^{(0)A}(\sigma) + \theta^i q_i^A(\sigma) + \frac{1}{2}\theta^i\theta^j q_{ij}^{(2)A}(\sigma)$, which contains q^A as a component. A superfield containing p_I is $\boldsymbol{p}_I(\sigma,\theta) \in \Gamma(T^*[1]\Sigma_2 \otimes \boldsymbol{x}^*(T^*[2]M))$, $\boldsymbol{p}_I(\sigma,\theta) = p_I^{(0)}(\sigma) + \theta^i p_{Ii}^{(1)}(\sigma) + \frac{1}{2}\theta^i\theta^j p_{Iij}(\sigma)$. A graded symplectic form $\boldsymbol{\Omega}$ of degree 1 is defined from Ω as

$$\boldsymbol{\Omega} = \int_{T[1]\Sigma_2} d^2\sigma d^2\theta\ \Phi^*\Omega = \int_{T[1]\Sigma_2} d^2\sigma d^2\theta (\delta\boldsymbol{x}^I \wedge \delta\boldsymbol{\xi}_I + k_{AB}\delta\boldsymbol{q}^A \wedge \delta\boldsymbol{\eta}^B + \delta\boldsymbol{p}_I \wedge \delta\boldsymbol{\chi}^I), \quad (6.64)$$

where $\boldsymbol{\xi}_I(\sigma,\theta) \in \Gamma(T^*[1]\Sigma_2 \otimes \boldsymbol{x}^*(T^*[3]M))$, $\boldsymbol{\eta}^A(\sigma,\theta) \in \Gamma(T^*[1]\Sigma_2 \otimes \boldsymbol{x}^*(T^*[3]E[1]))$, and $\boldsymbol{\chi}^I(\sigma,\theta) \in \Gamma(T^*[1]\Sigma_2 \otimes \boldsymbol{x}^*(T^*[3]T^*[2]M))$ are 'canonical conjugates' of $\boldsymbol{x}^I$, $\boldsymbol{q}^A$ and $\boldsymbol{p}_I$ with respect to $\boldsymbol{\Omega}$. The graded Poisson brackets on 'canonical conjugates' are

$$\begin{aligned}
\{\boldsymbol{x}^I(\sigma,\theta), \boldsymbol{\xi}_J(\sigma',\theta')\} &= \delta^I{}_J\delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),\\
\{\boldsymbol{q}^A(\sigma,\theta), \boldsymbol{\eta}^B(\sigma',\theta')\} &= k^{AB}\delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),\\
\{\boldsymbol{p}_I(\sigma,\theta), \boldsymbol{\chi}^J(\sigma',\theta')\} &= \delta^J{}_I\delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),
\end{aligned} \quad (6.65)$$

where $\delta^2(\theta-\theta') = (\theta_1+\theta'_1)(\theta_2+\theta'_2)$.

A Q-structure Hamiltonian functional $\boldsymbol{\Theta}$ is induced from the Q-structure Hamiltonian function Θ on $T^*[3]T^*[2]E[1]$. $\boldsymbol{\Theta}$ is defined as follows:

$$\boldsymbol{\Theta} = \int_{T[1]\Sigma} d^2\sigma d^2\theta\ \Phi^*\Theta. \quad (6.66)$$

Since the integration shifts degree by 2, $(\mathrm{Map}(T[1]S^1, \mathcal{M}), \boldsymbol{\Omega}, \boldsymbol{\Theta})$ is a QP manifold of degree 1 and $\boldsymbol{\Omega}, \boldsymbol{\Theta}$ define a Poisson structure.

A Q-structure Hamiltonian function Θ is of degree 4. Since no background structure except for a closed 4-form H is considered, the local coordinate expression of the pullback of Θ is canonically taken as

$$\Phi^*\Theta = \boldsymbol{\chi}^I\boldsymbol{\xi}_I + \frac{1}{2}k_{AB}\boldsymbol{\eta}^A\boldsymbol{\eta}^B + \frac{1}{4!}H_{IJKL}(\boldsymbol{x})\boldsymbol{\chi}^I\boldsymbol{\chi}^J\boldsymbol{\chi}^K\boldsymbol{\chi}^L. \quad (6.67)$$

The derived brackets for fundamental superfields are

$$
\begin{aligned}
&\{\{\boldsymbol{x}^I(\sigma,\theta),\Theta\},\boldsymbol{p}_J(\sigma',\theta')\} = -\{\{\boldsymbol{p}_I(\sigma,,\theta),\Theta\},\boldsymbol{x}^J(\sigma',\theta')\} = \delta^I{}_J\delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),\\
&\{\{\boldsymbol{q}^A(\sigma,\theta),\Theta\},\boldsymbol{q}^B(\sigma',\theta')\} = -k^{AB}\delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),\\
&\{\{\boldsymbol{p}_I(\sigma,\theta),\Theta\},\boldsymbol{p}_J(\sigma',\theta')\} = -\frac{1}{2}H_{IJKL}\boldsymbol{\chi}^K\boldsymbol{\chi}^L\delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),
\end{aligned} \tag{6.68}
$$

These derive the Poisson brackets for canonical conjugates by the pullback of the embedding map $\widehat{j}$:

$$
\begin{aligned}
&\{x^I(\sigma,\theta),p_J(\sigma',\theta')\}_{P.B.} = \widehat{j}^*\{\{\boldsymbol{x}^I(\sigma,\theta),\Theta\},\boldsymbol{p}_J(\sigma',\theta')\},\\
&\{q^A(\sigma,\theta),q^B(\sigma',\theta')\}_{P.B.} = \widehat{j}^*\{\{\boldsymbol{q}^A(\sigma,\theta),\Theta\},\boldsymbol{q}^B(\sigma',\theta')\},\\
&\{p_I(\sigma),p_J(\sigma')\}_{P.B.} = \widehat{j}^*\{\{\boldsymbol{p}_I(\sigma,\theta),\Theta\},\boldsymbol{p}_J(\sigma',\theta')\},
\end{aligned} \tag{6.69}
$$

This is realized by the 'gauge fixings' of auxiliary fields, which are carried out after the calculations as

$$
\begin{aligned}
&x_i^{(1)I} = x_{ij}^{(2)I} = q^{(0)A} = q_{ij}^{(2)A} = p_I^{(0)} = p_{Ii}^{(1)} = \chi^{(0)I} = 0,\\
&\chi_i^{(1)I} = \partial_i x^I,\\
&\eta_{ij}^{(2)A} = \partial_i q_j^A - \partial_j q_i^A.
\end{aligned} \tag{6.70}
$$

Since functions on the supermanifold $C^\infty(T[1]\Sigma_2)$ can be identified to the exterior algebra $\bigwedge^\bullet T^*\Sigma_2$, the original Poisson bracket (6.59) is obtained.

$\widehat{j}$ maps generalized currents (6.60) to current super functions of degree $0, 1$ and 2 as

$$
\begin{aligned}
&\widehat{j}\ J_{0(f)} = \boldsymbol{J}_{0(f)}(\sigma,\theta) = f(\boldsymbol{x}(\sigma,\theta)),\\
&\widehat{j}\ J_{1(\alpha,u)} = \boldsymbol{J}_{1(\alpha,u)}(\sigma,\theta) = \alpha_I(\boldsymbol{x}(\sigma,\theta))\boldsymbol{\chi}^I(\sigma,\theta) + u_A(\boldsymbol{x}(\sigma,\theta))\boldsymbol{q}^A(\sigma,\theta),\\
&\widehat{j}\ J_{2(G,K,F,H,E)} = \boldsymbol{J}_{2(G,K,F,H,E)}(\sigma,\theta)\\
&\qquad = \Bigg(G^I(\boldsymbol{x}(\sigma,\theta))\boldsymbol{p}_I(\sigma,\theta) + K_A(\boldsymbol{x}(\sigma,\theta))\boldsymbol{\eta}^A(\sigma,\theta) + \frac{1}{2}F_{AB}(\boldsymbol{x}(\sigma,\theta))\boldsymbol{q}^A(\sigma,\theta)\boldsymbol{q}^B(\sigma,\theta)\\
&\qquad\qquad + \frac{1}{2}B_{IJ}(\boldsymbol{x}(\sigma,\theta))\boldsymbol{\chi}^I(\sigma,\theta)\boldsymbol{\chi}^J(\sigma,\theta) + E_{AI}(\boldsymbol{x}(\sigma,\theta))\boldsymbol{\chi}^I(\sigma,\theta)\boldsymbol{q}^A(\sigma,\theta)\Bigg).
\end{aligned} \tag{6.71}
$$

Straightforward calculations show that the derived brackets describe the correct commutation

relations (6.62) of current algebras:

$$
\begin{aligned}
&\{\{\boldsymbol{J}_{0(f)},\boldsymbol{\Theta}\},\boldsymbol{J}_{0(f')}\}=0,\\
&\{\{\boldsymbol{J}_{1(u,\alpha)},\boldsymbol{\Theta}\},\boldsymbol{J}_{0(f')}\}=0,\\
&\{\{\boldsymbol{J}_{2(G,K,F,H,E)},\boldsymbol{\Theta}\},\boldsymbol{J}_{0(f')}\}=-G^I\frac{\partial f'}{\partial \boldsymbol{x}^I}\delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),\\
&\{\{\boldsymbol{J}_{1(u,\alpha)},\boldsymbol{\Theta}\},\boldsymbol{J}_{1(u',\alpha')}\}=-k^{AB}u_A u'_B\delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),\\
&\{\{\boldsymbol{J}_{2(G,K,F,B,E)},\boldsymbol{\Theta}\},\boldsymbol{J}_{1(u',\alpha')}\}=-\boldsymbol{J}_{1(\bar{u},\bar{\alpha})}\delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),\\
&\{\{\boldsymbol{J}_{2(G,K,F,B,E)},\boldsymbol{\Theta}\},\boldsymbol{J}_{2(G',K',F',B',E')}\}=-\boldsymbol{J}_{2(\bar{G},\bar{K},\bar{F},\bar{B},\bar{E})}\delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),
\end{aligned}
\tag{6.72}
$$

where $\bar{\alpha},\bar{u},\bar{G},\bar{K},\bar{F},\bar{H},\bar{E}$ are in Eq. (6.63).

The commutators of current functions with respect to the P-structure are

$$
\begin{aligned}
&\{\boldsymbol{J}_{0(f)},\boldsymbol{J}_{0(f')}\}=0,\quad \{\boldsymbol{J}_{1(u,\alpha)},\boldsymbol{J}_{0(f')}\}=0,\quad \{\boldsymbol{J}_{2(G,K,F,B,E)},\boldsymbol{J}_{0(f')}\}=0,\\
&\{\boldsymbol{J}_{1(u,\alpha)},\boldsymbol{J}_{1(u',\alpha')}\}=0,\\
&\{\boldsymbol{J}_{2(G,K,F,B,E)},\boldsymbol{J}_{1(u',\alpha')}\}=(G^I\alpha'_I-k^{AB}K_A u'_B)\delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),\\
&\{\boldsymbol{J}_{2(G,K,F,B,E)},\boldsymbol{J}_{2(G',K',F',B',E')}\}=\big[(G^J B'_{JI}+G'^J B_{JI}+k^{AB}(K_A E'_{BI}+E_{AI}K'_B))\boldsymbol{\chi}^I\\
&\qquad +(G^I E'_{AI}+G'^I E_{AI}+k^{BC}(K_B F'_{AC}+F_{AC}K'_B))\boldsymbol{q}^A\big]\,\delta^2(\sigma-\sigma')\delta^2(\theta-\theta'),
\end{aligned}
\tag{6.73}
$$

which derive the correct coefficients of anomaly terms in (6.62).

We have obtained the following result.

Theorem 6.1 *A current algebra* (6.62) *in three dimensions has a realization as a QP manifold of degree* 3 *on* $T^*[3](T^*[2]E[1])$*, i.e. a Lie algebroid up to homotopy on the vector bundle* T^*E*. The anomaly cancellation condition is equivalent to the condition that currents super functions are commutative under the Q-structure.*

7 QP Structures of Current Algebras in n Dimensions

We have seen that there are correspondences between generalized current algebras and QP manifolds in two and three dimensions. Generalizations to n dimensions is straightforward. A generalized current algebra in n dimensional worldvolume has a structure of a QP manifold of degree n.

7.1 Current Algebras in n Dimensions

Let $X_n = \Sigma_{n-1} \times \boldsymbol{R}$ be a manifold in n dimensions and a target space is a symplectic manifold in d dimensions. $\boldsymbol{R}$ is the time direction and Σ_{n-1} is the $n-1$ dimensional space. Let $q_i^{A^{(i)}}(\sigma)$ be a canonical quantity, which is a i-form, where $i = 0, 1, \cdots, n-1$. The space of $q_i^{A^{(i)}}(\sigma)$ is $\mathrm{Map}(\Sigma_{n-1}, T^*E)$, where $E = \bigoplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} E_i$ is a vector bundle on M, where $x^{A^{(0)}} = q_0^{A^{(0)}}$ is a map from Σ_{n-1} to M and $q_i^{A^{(i)}}$ is a section of $\wedge^i T^*\Sigma_{n-1} \oplus x^*(E_i)$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. The mass dimensions are $\dim[\sigma] = -1$ and $\dim[\partial] = 1$. $q_i^{A^{(i)}}(\sigma)$ has $\dim[q_i^{A^{(i)}}] = i$ in order to make the symplectic form ω dimensionless.

Let us consider the Poisson brackets for canonical conjugates:

$$\begin{aligned}
\{q_i^{A_1^{(i)}}(\sigma), q_j^{A_2^{(j)}}(\sigma')\}_{P.B.} &= (-1)^{ij+1}\{q_j^{A_2^{(j)}}(\sigma), q_i^{A_1^{(i)}}(\sigma')\}_{P.B.} \\
&= (-1)^i k_i^{A_1^{(i)} A_2^{(j)}} \delta_{i,n-j-1} \delta^{n-1}(\sigma - \sigma') \quad \text{for} \quad i \leq j, \qquad (7.74)
\end{aligned}$$

$$\begin{aligned}
&\{q_{n-1}^{A_1^{(0)}}(\sigma), q_{n-1}^{A_2^{(0)}}(\sigma')\}_{P.B.} \\
&\quad = -\frac{1}{(n-1)!} H^{A_1^{(0)} A_2^{(0)}}{}_{A_3^{(0)} \cdots A_{n+1}^{(0)}} dq_0^{A_3^{(0)}} \wedge \cdots \wedge dq_0^{A_{n+1}^{(0)}} \delta^{n-1}(\sigma - \sigma'), \qquad (7.75)
\end{aligned}$$

and the other commutation relations are 0, where H is a closed $n+1$-form and $k_i^{A_1^{(i)} A_2^{(i)}} = (-1)^{(i+1)(n-i)} k_i^{A_2^{(j)} A_1^{(i)}}$ is a metric on the subspace of $q_i^{A_1^{(i)}}$ and $q_j^{A_2^{(j)}}$ and $A^{(i)} = A^{(n-i-1)}$. Note that if n is odd, the $i = j$ term

$$\{q_m^{A_1^{(m)}}(\sigma), q_m^{A_2^{(m)}}(\sigma')\} = (-1)^m k_m^{A_1^{(m)} A_2^{(m)}} \delta^{n-1}(\sigma - \sigma'), \qquad (7.76)$$

is nonzero, where $m = \frac{n-1}{2}$.

Currents $J_l(q_i^{A^{(i)}}(\sigma), d_\sigma q_i^{A^{(i)}}(\sigma))$ of mass dimensions l are considered, where $l = 0, 1, \cdots, n-1$. The Poisson bracket of currents $\{J_l(q_i^{A^{(i)}}(\sigma), d_\sigma q_i^{A^{(i)}}(\sigma)), J_{l'}(q_i^{A^{(i)}}(\sigma'), d_{\sigma'} q_i^{A^{(i)}}(\sigma'))\}_{P.B.}$ has two terms, a commutator term and an anomaly term.

7.2 QP-structures

The space of worldvolume Σ_{n-1} is extended to a supermanifold $T[1]\Sigma_{n-1}$ with a local coordinate (σ^i, θ^i). A canonical quantity $q_i^{A^{(i)}}$ is extended to a superfield $\boldsymbol{q}_i^{A^{(i)}}$. A superfield $\boldsymbol{q}_i^{A^{(i)}}$ of degree i $(i = 0, \cdots, n-1)$ contains an original field $q_i^{A^{(i)}}$ as the i-th part. The target vector bundle is extended to the double graded vector bundle $\mathcal{M} = T^*[n]\left(T^*[n-1]\left(\bigoplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} E_i[i]\right)\right)$,

which is a QP-manifold of degree n with a QP structure (Ω, Θ). Note that the original canonical commutation relations of the original canonical quantities $q_i^{A^{(i)}}$ are not assumed in this stage.

The original target space T^*E is canonically embedded by the graded symplectic realization $j : T^*E \longrightarrow \mathcal{M}$. This induces a map $\widehat{j}$ on a mapping space $\widehat{j} : (q_i^{A_1^{(i)}}, 0, d_\sigma q_i^{A_1^{(i)}}) \longmapsto (\boldsymbol{q}_i^{A_1^{(i)}}, \boldsymbol{\eta}_n^{A_1^{(n-1)}}, \boldsymbol{\eta}_{i+1}^{A_1^{(i)}})$, where $\boldsymbol{\eta}_{i+1}^{A_1^{(i)}} \equiv (-1)^{i(n-i)} k_i^{A_1^{(i)} A_2^{(i)}} \boldsymbol{\eta}_{A_2^{(i)}\, i+1}$. Then the total space of a 'super' current algebras is $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{M})$. Superfields $\boldsymbol{\eta}_{A^{(i)}\, i+1}$ of degree $i+1$ derived from fiber local coordinates are introduced ($i = 0, \cdots, n-1$) as 'canonical conjugates' and the odd symplectic structure (P-structure) is defined from graded symplectic structure Ω of degree n on $\mathcal{M}$ as

$$\boldsymbol{\Omega} = \int_{T[1]\Sigma_{n-1}} d^{n-1}\sigma d^{n-1}\theta\ \Phi^*\Omega = \int_{T[1]\Sigma_{n-1}} d^{n-1}\sigma d^{n-1}\theta\ \left(\sum_{i=0}^{n-1} \delta \boldsymbol{q}_i^{A^{(i)}} \wedge \delta \boldsymbol{\eta}_{A^{(n-i-1)}\, n-i} \right), \quad (7.77)$$

where $A^{(i)} = A^{(n-i-1)}$. Thus the graded Poisson brackets for 'canonical conjugates' are

$$\begin{aligned} \{\boldsymbol{q}_i^{A_1^{(i)}}(\sigma,\theta), \boldsymbol{\eta}_{A_2^{(j)}\, j+1}(\sigma',\theta')\} &= -(-1)^{i(j+1)} \{\boldsymbol{\eta}_{A_2^{(j)}\, j+1}(\sigma,\theta), \boldsymbol{q}_i^{A_1^{(i)}}(\sigma',\theta')\} \\ &= \delta_{i,n-j-1} \delta^{A_1^{(i)}}{}_{A_2^{(i)}} \delta^{n-1}(\sigma-\sigma')\delta^{n-1}(\theta-\theta'), \end{aligned} \quad (7.78)$$

and the others are 0.

Let us take a Q-structure Hamiltonian function Θ on a QP-manifold of degree n, $\mathcal{M}$. The Q-structure functional on $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{M})$ is defined as

$$\boldsymbol{\Theta} = \int_{T[1]\Sigma_{n-1}} d^{n-1}\sigma d^{n-1}\theta\ \Phi^*\Theta. \quad (7.79)$$

This is of degree 2 and a Poisson bivector. Since $(\boldsymbol{\Omega}, \boldsymbol{\Theta})$ define a QP structure of degree 1, a Poisson structure is defined.

From the assumption that there is no background structure except for a closed $n+1$ form H, the Q-structure Hamilton function Θ of degree $n+1$ is uniquely determined as

$$\Phi^*\Theta = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \left(k_i^{A_1^{(i)} A_2^{(i)}} \boldsymbol{\eta}_{A_2^{(i)}\, i+1} \boldsymbol{\eta}_{A_1^{(i)}\, n-i} + \frac{1}{(n+1)!} H^{A_1^{(0)} \cdots A_{n+1}^{(0)}} \boldsymbol{\eta}_{A_1^{(0)}\, 1} \cdots \boldsymbol{\eta}_{A_{n+1}^{(0)}\, 1} \right), \quad (7.80)$$

where $\lfloor k \rfloor$ is the floor function which gives the largest integer less than or equal to k. The master equation of a Q-structure, $\{\Theta, \Theta\} = 0$, is satisfied if and only if H is a closed. where

$\Phi \in \mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{M})$. The derived bracket with respect to $\boldsymbol{\Theta}$ derives the correct canonical commutation relations:

$$\begin{aligned}
\{q_i^{A_1^{(i)}}(\sigma,\theta), q_j^{A_2^{(j)}}(\sigma',\theta')\}_{P.B.} &= \widehat{j}^*\{\{\boldsymbol{q}_i^{A_1^{(i)}}(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{q}_j^{A_2^{(j)}}(\sigma',\theta')\}, \\
\{q_{n-1}^{A_1^{(0)}}(\sigma,\theta), q_{n-1}^{A_2^{(0)}}(\sigma',\theta')\}_{P.B.} &= \widehat{j}^*\{\{\boldsymbol{q}_{n-1}^{A_1^{(0)}}(\sigma,\theta), \boldsymbol{\Theta}\}, \boldsymbol{q}_{n-1}^{A_2^{(0)}}(\sigma',\theta')\},
\end{aligned} \tag{7.81}$$

if the θ^{i+1} part of the superfields $\boldsymbol{\eta}_{i+1}^{A_1^{(i)}} \equiv (-1)^{i(n-i)} k_i^{A_1^{(i)} A_2^{(i)}} \boldsymbol{\eta}_{A_2^{(i)}\, i+1}$ is gauge fixed to $d_\sigma q_i^{A_1^{(i)}}$ and other auxiliary fields are projected by the pullback $\widehat{j}^*$ after the calculation.

For example, the commutation relation of two terms $f_{A_1^{(i)}}(x) d_\sigma q_i^{A_1^{(i)}}$ and $g(x(\sigma'))q_j^{A_2^{(j)}}$ on the original symplectic space is

$$\begin{aligned}
&\{f(x(\sigma)) d_\sigma q_i^{A_1^{(i)}}(\sigma), g(x(\sigma'))q_j^{A_2^{(j)}}(\sigma')\}_{P.B.} \\
&= -\left((-1)^i \delta_{i,n-j-1} k_i^{A_1^{(i)} A_2^{(i)}} g df - \delta_{i,0}\delta_{j,n-1} k_0^{A^{(0)} A_2^{(0)}} \frac{\partial f}{\partial x^{A^{(0)}}} g(x) dx_0^{A_1^{(0)}}\right) \delta^{n-1}(\sigma - \sigma') \\
&\quad + \left((-1)^i k_i^{A_1^{(i)} A_2^{(i)}} \delta_{i,n-j-1} f g\right)(\sigma') d\delta^{n-1}(\sigma-\sigma'),
\end{aligned} \tag{7.82}$$

for $i \leq j$ $(i \neq n-1)$, where $x^{A^{(0)}} = q_0^{A^{(0)}}$ is a canonical quantity of mass dimension zero. Since $f_{A_1^{(i)}}(x) d_\sigma q_i^{A_1^{(i)}}$ and $g(x(\sigma'))q_j^{A_2^{(j)}}$ are contained in superfields $f_{A_1^{(i)}}(\boldsymbol{x}(\sigma,\theta))\boldsymbol{\eta}_{i+1}^{A_1^{(i)}}(\sigma,\theta)$ and $g(\boldsymbol{x}(\sigma',\theta'))\boldsymbol{q}_j^{A_2^{(j)}}(\sigma',\theta')$, (7.82) are calculated by the derived bracket on the QP manifold:

$$\begin{aligned}
&\{\{f(\boldsymbol{x}(\sigma,\theta))\boldsymbol{\eta}_{i+1}^{A_1^{(i)}}(\sigma,\theta), \boldsymbol{\Theta}\}, g(\boldsymbol{x}(\sigma',\theta'))\boldsymbol{q}_j^{A_2^{(j)}}(\sigma',\theta')\} \\
&\qquad = -\Bigg((-1)^i \delta_{i,n-j-1} k_i^{A_1^{(i)} A_2^{(i)}} g \frac{\partial f}{\partial \boldsymbol{x}^{A^{(0)}}} \boldsymbol{\eta}_1^{A^{(0)}} \\
&\qquad\qquad - \delta_{i,0}\delta_{j,n-1} k_0^{A^{(0)} A_2^{(0)}} \frac{\partial f}{\partial x^{A^{(0)}}} \boldsymbol{\eta}_1^{A_1^{(0)}}\Bigg) \delta^{n-1}(\sigma-\sigma')\delta^{n-1}(\theta-\theta'),
\end{aligned} \tag{7.83}$$

for $i \leq j$ $(i \neq n-1)$, which derives the correct first term in (7.82). The commutation relation with respect to the graded Poisson bracket

$$\begin{aligned}
&\{f_{A_1^{(i)}}(\boldsymbol{x}(\sigma,\theta))\boldsymbol{\eta}_{i+1}^{A_1^{(i)}}(\sigma,\theta), g_{A_2^{(j)}}(\boldsymbol{x}(\sigma',\theta'))\boldsymbol{q}_j^{A_2^{(j)}}(\sigma',\theta')\} \\
&\qquad = (-1)^n f_{A_1^{(i)}} g_{A_2^{(i)}} k_i^{A_1^{(i)} A_2^{(i)}} \delta_{i,n-j-1} \delta^{n-1}(\sigma-\sigma')\delta^{n-1}(\theta-\theta'),
\end{aligned} \tag{7.84}$$

coincides with the correct coefficient of the second anomaly term in (7.82) if the systematic factor $(-1)^{i-n}$ is multiplied. We can find that the correct terms are calculated for more general terms. For example, for $f_{A_1^{(i)}\cdots A_l^{(j)}}(x) d_\sigma q_i^{A_1^{(i)}} \cdots d_\sigma q_j^{A_l^{(j)}}$, the graded Poisson brackets

for a superfields $f_{A_1^{(i)}\cdots A_l^{(j)}}(\boldsymbol{x})\boldsymbol{\eta}_{i+1}^{A_1^{(i)}}\cdots\boldsymbol{\eta}_{j+1}^{A_l^{(j)}}$ derives the correct commutation relations after the pullback $\widehat{j}^*\boldsymbol{\eta}_{i+1}^{(i+1)A^{(i)}} = d_\sigma q_i^{A^{(i)}}$.

Discussion in this section are summarized as follows. Let $J_{k(\boldsymbol{J}_k)}(q_i^{A^{(i)}}(\sigma), d_\sigma q_i^{A^{(i)}}(\sigma))$ be a current of mass dimension k. The space of canonical quantities is extended to a graded manifold, which is a QP manifold of degree n. Then this current is mapped to a super function $\widehat{j}: J_{k(\boldsymbol{J}_k)}(q_i^{A^{(i)}}(\sigma), d_\sigma q_i^{A^{(i)}}(\sigma)) \longmapsto \boldsymbol{J}_k(\boldsymbol{q}_i^{A^{(i)}}(\sigma,\theta), \boldsymbol{\eta}_i^{A^{(i)}}(\sigma,\theta))$ of degree k. The original commutation relation is calculated as

$$\begin{aligned}\{J_{k(\boldsymbol{J}_k)}(\sigma), J_{l(\boldsymbol{J}_l)}(\sigma')\}_{P.B.} &= -J_{k+l+1-n([\boldsymbol{J}_k,\boldsymbol{J}_l])}\delta^{n-1}(\sigma-\sigma') \\ &\quad +\langle J_k, J_k\rangle(\sigma')d_\sigma\delta^{n-1}(\sigma-\sigma'),\end{aligned} \tag{7.85}$$

where

$$[\boldsymbol{J}_k, \boldsymbol{J}_l] = \{\{\boldsymbol{J}_k(\sigma), \Theta\}, \boldsymbol{J}_l(\sigma')\}, \tag{7.86}$$

and $\langle J_k, J_l\rangle$ is obtained as

$$\widehat{j}^*\{\boldsymbol{J}_k(\sigma,\theta), \boldsymbol{J}_l(\sigma',\theta')\} = (-1)^{k-n}\langle J_k, J_l\rangle(\sigma')\delta^{n-1}(\sigma-\sigma')\delta^{n-1}(\theta-\theta'). \tag{7.87}$$

We have obtained the following theorem.

Theorem 7.1 *A generalized current algebra in n dimensions has a structures of a QP manifold of degree n. The anomalies cancel if and only if current super functions are commutative under the Q-structure.*

Let $\mathcal{L}$ be a maximal subalgebra of $C^\infty(\mathcal{M})$, such that elements are commutative $\{\boldsymbol{J}_k, \boldsymbol{J}_l\} = 0$ under the graded Poisson bracket and closed under the derived bracket $\{\{\boldsymbol{J}_k, \Theta\}, \boldsymbol{J}_l\} \in \mathcal{L}$. The anomaly cancellation condition is that a set of current functions is $\mathcal{L}$, which is a generalization of the Dirac structure. In fact, if a target space is a QP manifold $T^*[n]E[1]$, then a QP structure defines a generalized Courant-Dorfman bracket on $\Gamma E \oplus \wedge^{n-1}E^*$ from Example 4.9. In this example, the anomaly cancellation condition is equivalent to the restriction of a generalized Dirac structure L with respect to the generalized Courant-Dorfman bracket.

Since currents J_{n-1} of dimensions $n-1$ correspond to super functions $\boldsymbol{J}_{n-1}$ of degree $n-1$, the J_{n-1} part of current algebras has a structure of the Loday algebroid from the theorem 4.4.

8 Conclusions and Discussion

We have investigated generalized current algebras in any dimension. Symplectic structures of canonical conjugates have been reformulated by a QP manifold. Current algebras and anomalies in n dimensions have structures of an algebroid characterized by a QP manifold of degree n. The anomaly cancellation conditions are equivalent to the condition that current functions on a QP manifold consist of a commutative subalgebra.

A QP manifold structure of degree n of current algebras in n dimensions suggests a holographic correspondence of them with a quantum field theory in $n+1$ dimensions. Because a topological field theory in $n+1$ dimensions has a structure of a QP manifold of degree n via the AKSZ construction. These will be a generalization of the correspondence of the WZW model in two dimensions to the Chern-Simons gauge theory in three dimensions.

More generalizations of current algebras have been analyzed in two dimensions in [3]. Our discussion can be extended to that case and these current algebras will be reconstructed in terms of a QP manifold and generalized to higher dimensions. Our results will be generalized to a manifold with boundary. This case is connected to membrane theories, such as D-branes and M-branes.

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A Appendix

A.1 Current Algebras in Three Dimensions

Current algebras in Eqs. (6.62) and (6.63) calculated as components are following:

$$\{J_{0(f)}(\sigma), J_{0(f')}(\sigma')\}_{P.B.} = 0,$$

$$\{J_{1i(\alpha,u)}(\sigma), J_{0(f')}(\sigma')\}_{P.B.} = 0,$$

$$\{J_{2ij(G,K,F,H,E)}(\sigma), J_{0(f')}(\sigma')\}_{P.B.} = -\epsilon_{ij} G^I(x(\sigma)) \frac{\partial f'(x(\sigma))}{\partial x^I} \delta^2(\sigma - \sigma'),$$

$$\{J_{1i(\alpha,u)}(\sigma), J_{1j(\alpha',u')}(\sigma')\}_{P.B.} = \epsilon_{ij} u_A(x(\sigma)) u'_B(x(\sigma)) k^{AB} \delta^2(\sigma - \sigma'),$$

$$
\begin{aligned}
&\{J_{2ij(G,K,F,B,E)}(\sigma), J_{1k(\alpha',u')}(\sigma')\}_{P.B.} \\
&\qquad = -\epsilon_{ij} J_{1k(\bar{\alpha},\bar{u})}\delta^2(\sigma-\sigma') + \epsilon_{ij}(-k^{AB}u'_A K_B + \alpha'_I G^I)(x(\sigma'))\partial_k \delta^2(\sigma-\sigma'), \\
&\{J_{2ij(G,K,F,B,E)}(\sigma), J_{2kl(G',K',F',B',E')}(\sigma')\}_{P.B.} \\
&\qquad = -\epsilon_{ij}\epsilon_{kl} J_{2(\bar{G},\bar{K},\bar{F},\bar{B},\bar{E})}\delta^2(\sigma-\sigma') \\
&\qquad\quad -\epsilon_{ij}\epsilon_{kl}\left(G'^I B_{IJ} + G^I B'_{IJ} + k^{AB}(E_{AJ}K'_B + E'_{AJ}K_B)\right)\epsilon^{mn}(\partial_m x^J \partial_n)\delta^2(\sigma-\sigma') \\
&\qquad\quad -\epsilon_{ij}\epsilon_{kl}\left(G'^I E_{AI} + G^I E'_{AI} + k^{BC}(K_C F'_{AB} + K'_C F_{AB})\right)\epsilon^{mn}(q_m^A \partial_n)\delta^2(\sigma-\sigma'), \quad \text{(A.88)}
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{\alpha}_I &= G^J \frac{\partial \alpha'_I}{\partial x^J} + \alpha'_J \frac{\partial G^J}{\partial x^I} + k^{AB}\left(-u'_A \frac{\partial K_B}{\partial x^I} + u'_A E_{BI}\right), \\
\bar{u}_A &= G^I \frac{\partial u'_A}{\partial x^I} - k^{BC} u'_B F_{CA},
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{G}^I &= G^J \frac{\partial G'^I}{\partial x^J} - G'^J \frac{\partial G^I}{\partial x^J}, \\
\bar{K}_A &= G^I \frac{\partial K'_A}{\partial x^I} - G'^I \frac{\partial K_A}{\partial x^I} + k^{BC} K'_B F_{AC} + E_{AI} G'^I, \\
\bar{F}_{AB} &= G^I \frac{\partial F'_{AB}}{\partial x^I} - G'^I \frac{\partial F_{AB}}{\partial x^I} + k^{CD}(F_{AC}F'_{DB} - F_{BC}F'_{DA}),
\end{aligned}
$$

$$
\begin{aligned}
\bar{B}_{IJ} &= G^K \frac{\partial B'_{IJ}}{\partial x^K} - \frac{\partial G^K}{\partial x^I} B'_{JK} - \frac{\partial G^K}{\partial x^J} B'_{KI} - G'^K \left(\frac{\partial B_{IJ}}{\partial x^K} + \frac{\partial B_{JK}}{\partial x^I} + \frac{\partial B_{KI}}{\partial x^J}\right) + G^K G'^L H_{KLIJ} \\
&\quad + k^{AB}\left((E_{AJ}E'_{BI} - E_{AI}E'_{BJ}) + K'_B\left(\frac{\partial E_{AJ}}{\partial x^I} - \frac{\partial E_{AI}}{\partial x^J}\right) + \left(E'_{BJ}\frac{\partial K_A}{\partial x^I} - E'_{BI}\frac{\partial K_A}{\partial x^J}\right)\right), \\
\bar{E}_{AI} &= G^J \frac{\partial E'_{AI}}{\partial x^J} + \frac{\partial G^J}{\partial x^I} E'_{AJ} + G'^J \left(\frac{\partial E_{AJ}}{\partial x^I} - \frac{\partial E_{AI}}{\partial x^J}\right) \\
&\qquad + k^{BC}\left(E_{BI}F'_{CA} - F_{BA}E'_{CI} + \frac{\partial F_{AB}}{\partial x^I} K'_C - \frac{\partial K_B}{\partial x^I} F'_{CA}\right). \qquad \text{(A.89)}
\end{aligned}
$$

Anomaly cancellation conditions are given by

$$
\begin{aligned}
&-k^{AB}u'_A K_B + \alpha'_I G^I = 0, \\
&G'^I B_{IJ} + G^I B'_{IJ} + k^{AB}(E_{AJ}K'_B + E'_{AJ}K_B) = 0, \\
&G'^I E_{AI} + G^I E'_{AI} + k^{BC}(K_C F'_{AB} + K'_C F_{AB}) = 0. \qquad \text{(A.90)}
\end{aligned}
$$

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